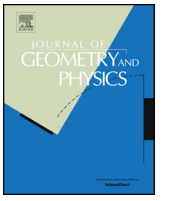
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A geometric model for non-uniform processes of morphogenesis



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ABSTRACT

In this paper we present an application of the groupoid theory to the study of relevant case of material evolution phenomena, the *process of morphogenesis*. Our theory is inspired by Walter Noll's theories of continuous distributions and provides a unifying and very simple framework of these phenomena. We present the explicit equation, the *morphogenesis equation*, to calculate the *material distributions* associated to this phenomenon.

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1. Introduction

In *Continuum Mechanics* the distribution of constitutive properties reflects the material structure of the body. This distribution measures the response of the material to a deformation in a local neighbourhood at each particle. In 1954 the W. Noll's thesis, entitled "*On the Continuity of the Solid and Fluid States*" [26], established a new approach of uniformity and homogeneity of elastic materials. This approach is the physical basis of our paper and it supports on the use of the notion of *material isomorphism*, and the differentiable map called *mechanical response* (see also [25,27,33,37].

The notion of groupoid is intimately linked to the study of materials, and in fact, in the works of Walter Noll one can implicitly recognize the existence of this algebraic structure when considering the collection of all material isomorphisms obtained through a given constitutive law.

For a simple material $\mathcal{B}$, the collection of the material isomorphisms constitutes the so-called material groupoid, $\Omega(\mathcal{B})$, which is in fact a subgroupoid of the Lie groupoid formed by all the linear isomorphisms between the tangent spaces of all the pairs of points of the material body [15,20]. This is the so-called *material groupoid* and it is not necessarily a Lie groupoid. This lack of differentiability measures the lack of differentiable uniformity of the material (for explicit examples see [18,21]).

On the other hand, a natural question is what infinitesimal properties of the material groupoid reflect inhomogeneity. If it is a Lie subgroupoid, we can associate with it the corresponding Lie algebroid and characterize the homogeneity by its properties (in short, it is a global way of dealing with material G-structures that had been considered in other approaches [3,4,10,11]). But even if the material groupoid is not Lie, we have been able to construct a generalization of the associated Lie algebroid, which we call *characteristic distribution* and which is still able to give answers to the concepts of homogeneity and integrability [5].

This distribution is integrable (in the sense of Stefan and Sussmann [31,32]) and determines generalized foliations, one in the subgroupoid and the other in the body manifold, so that, roughly speaking, differentiability is introduced where there was none, and the leaves in the material groupoid become true Lie groupoids on the leaves in the base on which they project.

We are now interested in a new concept, the possible evolution of the material. To address this issue, we need to consider constitutive laws that take into account time (or any other evolutionary parameter(s)), so that the material body $\mathcal{B}$ is replaced by the *space-time fibre bundle* $\mathcal{C} = \mathbb{R} \times \mathcal{B} \longrightarrow \mathbb{R}$, and more specifically, by the vertical fibre of this fibration, which is a vector subbundle of the tangent bundle $T\mathcal{C}$. $\mathcal{C}$ is also called the *body-time manifold*. So, via the evolution mechanical response $W : \mathcal{C} \times Gl(3, \mathbb{R}) \to V$ of the material, we can introduce the notion of *time-material isomorphism*, and the corresponding material groupoid $\Omega(\mathcal{C})$.

As opposed to the uniformity in the spatial case, new material properties arise, associated with the evolution of the body. In particular, the temporal counterpart of uniformity is a specific case of evolution of the material called *remodelling*. Intuitively, a material evolution presents a remodelling when the constitutive properties of the material do not change with the time. This kind of evolution may be found in biological tissues [29]. Wolff's law of trabecular architecture of bones is another relevant example [35]. Growth and resorption are given by an evolution material with volume increase or volume decrease of the material body $\mathcal{B}$. We will deal with particular cases of growth and resorption; those in which the material properties do not change.

We say that $\mathcal{C}$ is a process of *aging* if it is not a process of remodelling. Clearly, if the material response is not preserved along the time via material isomorphisms, then the constitutive properties are changing with the time. In [17,22], we use the corresponding characteristic distributions and their foliations to present several results characterizing the different types of remodelling and aging. Here, however, we are interested in a particular case of evolution: *morphogenesis*.

In order to work with the concept of morphogenesis, we will need to extend this notion to groupoids, introducing the notion of *normalizoid* of a subgroupoid in a groupoid. This generalization may be found in [2] to study uniformity and homogeneity in functional graded materials (FGM).

A material point X is said to undergo a process of evolution without morphogenesis when its symmetry groups at all the instants are conjugated. $\mathcal{C}$ is said to be a process of evolution without morphogenesis when all the points undergo a process of evolution without morphogenesis. Hence, remodelling is an example of the process of evolution without morphogenesis. In other words, as it is natural, any process of morphogenesis is a particular case of aging. Thus, a process of morphogenesis entails a breakdown of symmetry, a sudden change in the material symmetry type. Using the technique of characteristic distributions, we have been able to express it in terms of a differential equation, which we call the morphogenesis equation.

The paper is structured as follows. In section 2 we present an overview of the definitions and results needed in groupoid theory and their associated characteristic distributions. In section 3 we introduce the concept of material evolution of simple bodies, while in section 4 we construct the material groupoid associated with the vertical body-time manifold, next to the associated material distributions. Section 5 is devoted to discuss the phenomenon of morphogenesis. It is convenient to point out that the main development of this paper is contained in this section. First, we start studying a natural generalization of the notion "*normalizer*" and "*normal subgroup*" from groups to groupoids. Here, we prove some general results (Proposition 7, Proposition 8 and Corollary 9), but we do not do a deep study of this theory, because our interest is focused on its application to Continuum Mechanics. Then, we use this notion to deal with the notion of morphogenesis. In fact, not only theoretical constructions are made, but also the linear equations that determine the morphogenesis are obtained. In

particular, it is important to highlight that we give a specific way to construct the material distribution which characterizes the phenomenon of morphogenesis. Namely, the material distribution is pointwise generated by the left-invariant vector fields Θ on $\Phi(\mathcal{V})$, the space of linear isomorphisms between fibres of the vertical bundle $\mathcal{V}$ of the body-time manifold, satisfying that

$$TW([\Theta, \Lambda]) = 0,$$

for any left invariant admissible vector field Λ for the pair $(\Phi(\mathcal{V}), \Omega(\mathcal{C}))$ which is tangent to the α-fibres of $\Phi(\mathcal{V})$, which is defined by the elements g of $\Phi(\mathcal{V})$ in such a way that $\alpha(g) = x$ (see Definition 1 and Example 5). Here, $W : \Phi(\mathcal{V}) \to V$, is the evolution mechanical response of the material, and TW refers to the induced tangent of the map W (see Theorem 12).

This fact permits us to introduce the *morphogenesis equations*,

$$y^i_l \left(\Theta^l_r \Lambda^r_j + \Theta^k \frac{\partial \Lambda^l_j}{\partial x^k} + \lambda \frac{\partial \Lambda^l_j}{\partial t} - \Lambda^l_r \Theta^r_j \right) \frac{\partial W}{\partial y^i_j} = 0 \tag{1}$$

for any matrix function $\left(\Lambda^i_j\right)$ fulfilling the equation,

$$y^i_l \Lambda^l_j \frac{\partial W}{\partial y^i_j} = 0$$

Here the solutions are the functions λ, Θ^i, and Θ^j_i. We present analogous constructions for the evolution of a particle X. The above distribution, and its associated equation, are used to characterize the processes of morphogenesis in such a way that the dimension of the space of solutions will provide us the information about the existence of processes of morphogenesis in the evolution of the material.

2. An overview on groupoids and distribution

We will give here a very brief introduction on *(Lie) groupoids* and the relation with *(smooth) distributions* which is crucial to understand the results proved in this paper. For a detailed study we refer to [5] (see also [19,18]).

On the one hand, groupoids may be though as a natural generalization of the notion of group, and they were introduced in 1926 by Brandt [1]. Adding differential structure, we have the notion of *Lie groupoid* which is due to Ehresmann ([9,6–8]) and Pradines ([28]). We will follow the reference on groupoids [24] (see also [12], [38] or [36] for a more intuitive view).

Definition 1. Let M be a set. A *groupoid* over M is given by a set Γ equipped with the maps $\alpha, \beta : \Gamma \to M$ (*source map* and *target map* respectively), $\epsilon : M \to \Gamma$ (*section of identities*), $i : \Gamma \to \Gamma$ (*inversion map*) and $\cdot : \Gamma_{(2)} \to \Gamma$ (*composition law*). Here, $\Gamma_{(k)}$ denotes the k-tuples $(g_1, \ldots, g_k) \in \Gamma \overset{k)}{\times \ldots \times} \Gamma$ such that $\alpha(g_i) = \beta(g_{i+1})$ for $i = 1, \ldots, k-1$. The following properties are satisfied:

(1) α and β are surjective and for each $(g, h) \in \Gamma_{(2)}$,

$$\alpha(g \cdot h) = \alpha(h), \quad \beta(g \cdot h) = \beta(g).$$

(2) Associativity of the composition law, i.e.,

$$g \cdot (h \cdot k) = (g \cdot h) \cdot k, \ \forall (g, h, k) \in \Gamma_{(3)}.$$

(3) For all $g \in \Gamma$,

$$g \cdot \epsilon(\alpha(g)) = g = \epsilon(\beta(g)) \cdot g.$$

In particular,

$$\alpha \circ \epsilon \circ \alpha = \alpha, \quad \beta \circ \epsilon \circ \beta = \beta.$$

(4) For each $g \in \Gamma$,

$$i(g) \cdot g = \epsilon(\alpha(g)), \quad g \cdot i(g) = \epsilon(\beta(g)).$$

Then,

$$\alpha \circ i = \beta, \quad \beta \circ i = \alpha.$$

These maps (α, β, ϵ, i, and $\cdot$) will be called the *structure maps*. We will denote this groupoid by $\Gamma \rightrightarrows M$.

Observe that, since α and β are surjective we get

$$\alpha \circ \epsilon = Id_M, \quad \beta \circ \epsilon = Id_M,$$

where Id_M is the identity at M.

Sometimes M is denoted by $\Gamma_{(0)}$ and it is identified with the set $\epsilon(M)$ of identities of Γ. Γ is also denoted by $\Gamma_{(1)}$. The elements of M are called *objects* and the elements of Γ are called *morphisms*. The map $(\alpha, \beta) : \Gamma \to M \times M$ is called the *anchor map* and the space of sections of the anchor map is denoted by $\Gamma_{(\alpha,\beta)}(\Gamma)$. Finally, for each $g \in \Gamma$ the element $i(g)$ is denoted by g^{-1}.

One may think that the definition of groupoid looks too "*abstract*". However, roughly speaking, a groupoid may be depicted as a set of "*arrows*" (Γ) joining points (M), in such a way that any two arrows may be composed if the ending point of one coincides with the starting point of the other. Then, assuming natural conditions derived of the properties of a composition in a group, we get the definition of groupoid.

Definition 2. A *subgroupoid* of a groupoid $\Gamma \rightrightarrows M$ is a groupoid $\Gamma' \rightrightarrows M'$ such that $M' \subseteq M$, $\Gamma' \subseteq \Gamma$ and the structure maps of Γ' are the restrictions of the structure maps of Γ.

In particular, a subgroupoid has the same composition law of the corresponding groupoid.

Example 1. A group G is a groupoid over a point and the operation law of the groupoid, $\cdot$, is the operation in G.

Example 2. For any set X, the product space $X \times X$ is a groupoid over X, called the *pair groupoid*, in such a way that,

$$\alpha(a, b) = a, \quad \beta(a, b) = b, \ \forall (a, b) \in X \times X$$
$$(c, b) \cdot (a, c) = (a, b), \ \forall (c, b), (a, c) \in X \times X$$

Example 3. Let X be a set and G be a group. Then we can construct a groupoid $X \times X \times G \rightrightarrows X$ where the source map is the second projection, the target map is the third projection and the composition law is given by the composition in G, i.e.,

$$(t, y, g) \cdot (x, t, h) = (x, y, g \cdot h),$$

for all $(t, y, g), (x, t, h) \in X \times X \times G$. This groupoid is called *trivial groupoid on X with group G*.

Next, let us describe the crucial example of groupoid for the purpose of this paper.

Example 4. Let A be a vector bundle on a manifold M. Denote by A_z, the fibre of A over a $z \in M$. Then, the set $\Phi(A)$, consisting of all linear isomorphisms $L_{x,y} : A_x \to A_y$ for any $x, y \in M$, may be endowed with the structure of groupoid with structure maps,

(i) $\alpha\left(L_{x,y}\right) = x$
(ii) $\beta\left(L_{x,y}\right) = y$
(iii) $L_{y,z} \cdot G_{x,y} = L_{y,z} \circ G_{x,y}, \ L_{y,z} : A_y \to A_z, \ G_{x,y} : A_x \to A_y$

We will call this groupoid as the *frame groupoid on A*.

A particular relevant case is the *1-jets groupoid on M* and it arises when A is the tangent bundle TM of M. This groupoid is denoted by $\Pi^1(M, M)$. Notice that any isomorphism $L_{x,y} : T_xM \to T_yM$ may be written as a 1-jet $j^1_{x,y}\psi$ of a local diffeomorphism ψ from M to M. Notice that the 1-jet $j^1_{x,y}\psi$ may be identified with the tangent map $T_x\psi : T_xM \to T_yM$ (see [30] for details).

Definition 3. Let $\Gamma \rightrightarrows M$ be a groupoid with α and β the source map and target map, respectively. For each $x \in M$, the set

$$\Gamma^x_x = \beta^{-1}(x) \cap \alpha^{-1}(x),$$

is called the *isotropy group of* Γ at x. The set

$$\mathcal{O}(x) = \beta\left(\alpha^{-1}(x)\right) = \alpha\left(\beta^{-1}(x)\right),$$

is called the *orbit* of x, or *the orbit* of Γ through x.

Observe that the isotropy groups inherit a *bona fide* group structure.

Definition 4. If $\mathcal{O}(x) = M$ for all $x \in M$ (or equivalently $(\alpha, \beta) : \Gamma \to M \times M$ is a surjective map) the groupoid $\Gamma \rightrightarrows M$ is called *transitive*. The sets,

$$\alpha^{-1}(x) = \Gamma_x, \quad \beta^{-1}(x) = \Gamma^x,$$

are called *α-fibre at x* and *β-fibre at x*, respectively. We will denote

$$\Gamma_x^y = \Gamma_x \cap \Gamma^y,$$

for all $x, y \in M$.

Definition 5. Let $\Gamma \rightrightarrows M$ be a groupoid. We may define the *left translation by $g \in \Gamma$* as the map $L_g : \Gamma^{\alpha(g)} \to \Gamma^{\beta(g)}$, given by

$$h \mapsto g \cdot h.$$

We may define the *right translation* by g, $R_g : \Gamma_{\beta(g)} \to \Gamma_{\alpha(g)}$, analogously.

Note that, the identity map on Γ^x may be written as the following translation map,

$$Id_{\Gamma^x} = L_{\epsilon(x)}. \tag{2}$$

For any $g \in \Gamma$, the left (resp. right) translation on g, L_g (resp. R_g), is a bijective map with inverse $L_{g^{-1}}$ (resp. $R_{g^{-1}}$).

Definition 6. A *Lie groupoid* is a groupoid $\Gamma \rightrightarrows M$ such that Γ is a smooth manifold, M is a smooth manifold and the structure maps are smooth. Furthermore, the source and the target map are submersions.

A *Lie subgroupoid* of $\Gamma \rightrightarrows M$ is a Lie groupoid $\Gamma' \rightrightarrows M'$ such that it is a subgroupoid of Γ satisfying that Γ' and M' are submanifolds of Γ and M respectively.

Notice that the following statements are immediate:

- ϵ is an injective immersion.
- For each $g \in \Gamma$, the left translation L_g (resp. right translation R_g) is a diffeomorphism, for all $g \in \Gamma$.
- For each $k \in \mathbb{N}$, $\Gamma_{(k)}$ is a smooth manifold, for all $k \in \mathbb{N}$.
- The β-fibres and the α-fibres are closed submanifolds of Γ.

As first examples, any Lie group G (Example 1), the pair groupoid of a manifold M (Example 2), and any trivial groupoid on a manifold M with a Lie group G (Example 3) are Lie groupoids.

Example 5. The frame groupoid $\Phi(A)$ on a vector bundle A (Example 4) is a Lie groupoid. Let us consider two local coordinates, (x^i) and (y^j), on open neighbourhoods $U, V \subseteq M$, respectively, and two local bases of sections of A_U and A_V, $\{\alpha_p\}$ and $\{\beta_q\}$, respectively. The corresponding local coordinates $(x^i \circ \pi, \alpha^p)$ and $(y^j \circ \pi, \beta^q)$ on A_U and A_V are given by

- For any $a \in A_U$,
$$a = \alpha^p(a)\, \alpha_p\left(x^i(\pi(a))\right).$$
- For any $a \in A_V$,
$$a = \beta^q(a)\, \beta_q\left(y^j(\pi(a))\right).$$

Then, we can construct a local coordinate system on $\Phi(A)$

$$\Phi\left(A_{U,V}\right) : \left(x^i, y_i^j, y_i^j\right),$$

where, $A_{U,V} = \alpha^{-1}(U) \cap \beta^{-1}(V)$ and for each $L_{x,y} \in \alpha^{-1}(x) \cap \beta^{-1}(y) \subseteq \alpha^{-1}(U) \cap \beta^{-1}(V)$, we have

- $x^i\left(L_{x,y}\right) = x^i(x)$.
- $y^j\left(L_{x,y}\right) = y^j(y)$.
- $y_i^j\left(L_{x,y}\right) = A_{L_{x,y}}$, where $A_{L_{x,y}}$ is the associated matrix to the induced map of $L_{x,y}$ using the local coordinates $(x^i \circ \pi, \alpha^p)$ and $(y^j \circ \pi, \beta^q)$.

In the particular case of the 1-jets groupoid on M, $\Pi^1(M,M)$, the local coordinates will be denoted as follows

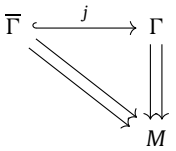
$$\Pi^1(U,V):\left(x^i,y^j,y^j_i\right), \tag{3}$$

where, for each $j^1_{x,y}\psi\in\Pi^1(U,V)$

- $x^i\left(j^1_{x,y}\psi\right)=x^i(x)$.
- $y^j\left(j^1_{x,y}\psi\right)=y^j(y)$.
- $y^j_i\left(j^1_{x,y}\psi\right)=\dfrac{\partial\left(y^j\circ\psi\right)}{\partial x^i_{|x}}$.

In the application to continuum mechanics, we will introduce the *material groupoid*, which will be defined as a subgroupoid of special cases of the frame groupoid. In particular, we will deal with the 1-jets groupoid $\Pi^1(\mathcal{B},\mathcal{B})$ on a manifold $\mathcal{B}$ (body) and a frame groupoid $\Phi(\mathcal{V})$ of the vertical bundle $\mathcal{V}$ of a given vector bundle $\mathcal{C}$ (material evolution).

From now on, we will deal with a (not necessarily a Lie) subgroupoid $\overline{\Gamma}\rightrightarrows M$ of a Lie groupoid $\Gamma\rightrightarrows M$,



where j is the inclusion map. We will also denote by $\overline{\alpha}$, $\overline{\beta}$, $\overline{\epsilon}$ and $\overline{i}$ the restrictions of the structure maps α, β, ϵ and i of Γ to $\overline{\Gamma}$.

In what follows, we will construct of the so-called *characteristic distribution* $A\overline{\Gamma}^T$ (see [5,19]).

Definition 7. An *admissible* (local) vector field $\Theta\in\mathfrak{X}_{loc}(\Gamma)$ on Γ for the pair $\left(\Gamma,\overline{\Gamma}\right)$ is a (local) vector field on Γ satisfying that,

(i) Θ is tangent to the β-fibres,

$$\Theta(g)\in T_g\beta^{-1}(\beta(g)),$$

for all g in the domain of Θ.

(ii) Θ is invariant by left translations,

$$\Theta(g)=T_{\epsilon(\alpha(g))}L_g\left(\Theta\left(\epsilon\left(\alpha(g)\right)\right)\right),$$

for all g in the domain of Θ.

(iii) The (local) flow φ^Θ_t of Θ satisfies

$$\varphi^\Theta_t(\epsilon(x))\in\overline{\Gamma},$$

for all $x\in M$.

In other words, an admissible vector field is a left invariant vector field on Γ whose flow at the identities is totally contained in $\overline{\Gamma}$.

Moreover, a vector field Θ of Γ satisfies conditions (i) and (ii) if, and only if, its local flow φ^Θ_t is left-invariant or, equivalently,

$$L_g\circ\varphi^\Theta_t=\varphi^\Theta_t\circ L_g,\ \forall g,t.$$

Therefore, condition (iii) is equivalent to the following one,

(iii)' The (local) flow φ^Θ_t of Θ at $\overline{g}$ is totally contained in $\overline{\Gamma}$, for all $\overline{g}\in\overline{\Gamma}$.

Hence, the admissible vector fields are the left-invariant vector fields on Γ whose integral curves are confined inside or outside $\overline{\Gamma}$.

The family of admissible vector fields for the pair $\left(\Gamma,\overline{\Gamma}\right)$ is denoted by $\mathcal{C}_{(\Gamma,\overline{\Gamma})}$, or simply, $\mathcal{C}$ if there is no danger of confusion.

Definition 8. The *characteristic distribution of* $\overline{\Gamma}$, denoted by $A\overline{\Gamma}^T$ is the smooth distribution on Γ linearly generated by the admissible (local) vector fields.

Namely, for each $g \in \Gamma$, the fibre at g is given by

$$A\overline{\Gamma}^T_g = Span\{\Theta(g) : \Theta \text{ is an admissible vector field}\}$$

Observe that, for all $g \in \Gamma$, the zero vector $0_g \in T_g\Gamma$ is contained in the fibre $A\overline{\Gamma}^T_g$ of the distribution at g (we remit to [19,21] for non trivial examples). Then, the distribution $A\overline{\Gamma}^T$ generated by the vector spaces $A\overline{\Gamma}^T_g$ is the *characteristic distribution of* $\overline{\Gamma}$.

Remark 1. This construction of the characteristic distribution associated to a subgroupoid $\overline{\Gamma}$ of a Lie groupoid Γ may be thought as a generalization of the construction of the associated Lie algebroid to a given Lie groupoid (see [24]). $\diamond$

The algebraic structure associated to a groupoid allows us to define more objects. Particularly, one of them is a smooth distribution over the base M denoted by $A\overline{\Gamma}^\sharp$. The other one is a "*differentiable*" correspondence $A\overline{\Gamma}$ which associates to any point x of M a vector subspace of $T_{\epsilon(x)}\Gamma$. Notice that, the differentiability of $A\overline{\Gamma}$ generalizes the differentiability of a vector bundle in the sense that $A\overline{\Gamma}$ is a (smooth) vector bundle if the distribution $A\overline{\Gamma}^T$ is regular (the dimension of the fibres does not varies). Both constructions are characterized by the following diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{A\overline{\Gamma}^T} & \mathcal{P}(T\Gamma) \\ {\scriptstyle\epsilon}\uparrow & \nearrow_{A\overline{\Gamma}} & \downarrow{\scriptstyle T\alpha} \\ M & \xrightarrow{A\overline{\Gamma}^\sharp} & \mathcal{P}(TM) \end{array}$$

where $\mathcal{P}(E)$ defines the power set of E. Therefore, for any $x \in M$, the fibres are given by,

$$A\overline{\Gamma}_x = A\overline{\Gamma}^T_{\epsilon(x)}$$
$$A\overline{\Gamma}^\sharp_x = T_{\epsilon(x)}\alpha\left(A\overline{\Gamma}_x\right)$$

The distribution $A\overline{\Gamma}^\sharp$ is called *base-characteristic distribution of* $\overline{\Gamma}$. Observe that, those distributions are possibly singular.

Notice that, taking into account that $A\overline{\Gamma}^T$ is locally generated by left-invariant vector fields, we have that for each $g \in \Gamma$,

$$A\overline{\Gamma}^T_g = T_{\epsilon(\alpha(g))}L_g\left(A\overline{\Gamma}^T_{\epsilon(\alpha(g))}\right),$$

i.e., the characteristic distribution is *left-invariant*.

Theorem 1 *([5,19]). Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $\overline{\Gamma}$ be a subgroupoid of Γ (not necessarily a Lie groupoid) over M. Then, the characteristic distribution $A\overline{\Gamma}^T$ is integrable and its associated foliation $\overline{\mathcal{F}}$ of Γ satisfies that $\overline{\Gamma}$ is a union of leaves of $\overline{\mathcal{F}}$.*

This result is a consequence of the celebrated Stefan-Sussman's theorem [31,32] which deals with the integrability of singular distributions. Each leaf at a point $g \in \Gamma$ will be denoted by $\overline{\mathcal{F}}(g)$ and the *characteristic foliation of* $\overline{\Gamma}$ will be given by the family of the leaves of $\overline{\mathcal{F}}$ at points of $\overline{\Gamma}$. The foliation $\overline{\mathcal{F}}$ satisfies that

(i) For any $g \in \Gamma$,

$$\overline{\mathcal{F}}(g) \subseteq \Gamma^{\beta(g)}.$$

Indeed, if $g \in \overline{\Gamma}$, then

$$\overline{\mathcal{F}}(g) \subseteq \overline{\Gamma}^{\beta(g)}.$$

(ii) *Left-invariance:* for any $g, h \in \Gamma$ such that $\alpha(g) = \beta(h)$, we have

$$\overline{\mathcal{F}}(g \cdot h) = g \cdot \overline{\mathcal{F}}(h).$$

It is important to point out that the leaves of the characteristic foliation cover $\overline{\Gamma}$ but, however, it is not exactly a foliation of $\overline{\Gamma}$ (because $\overline{\Gamma}$ is not necessarily a manifold).

As a summary, without any assumption of differentiability over $\overline{\Gamma}$, we have that $\overline{\Gamma}$ is union of leaves of a foliation of Γ. This provides some kind of "*differentiable*" structure over $\overline{\Gamma}$.

Notice that, analogously to Theorem 1, we may prove that the base-characteristic distribution $A\overline{\Gamma}^{\sharp}$ is integrable. Thus, we will denote the foliation which integrates the base-characteristic distribution over the base M by $\mathcal{F}$. For each point $x \in M$, the leaf of $\mathcal{F}$ containing x will be denoted by $\mathcal{F}(x)$. $\mathcal{F}$ will be called the *base-characteristic foliation of* $\overline{\Gamma}$.

Example 6. Let $\sim$ be an equivalence relation on a manifold M, i.e., a binary relation that is reflexive, symmetric and transitive. Then, define the subset $\mathcal{O}$ of $M \times M$ given by

$$\mathcal{O} := \{(x, y) \,:\, x \sim y\}. \tag{4}$$

Hence, $\mathcal{O}$ is a subgroupoid of $M \times M$ over M. In fact, this is equivalent to the properties reflexive, symmetric and transitive. For each $x \in M$, we denote by $\mathcal{O}_x$ to the orbit around x,

$$\mathcal{O}_x := \{y \,:\, x \sim y\}.$$

Notice that the orbits divide M into a disjoint union of subsets. However, these are not (necessarily) submanifolds.

On the other hand, the base-characteristic foliation gives us a foliation $\mathcal{F}$ of M such that

$$\mathcal{F}(x) \subseteq \mathcal{O}_x, \ \forall x \in M.$$

So, consider any arbitrary equivalence relation on a manifold M. Maybe the orbits are not manifolds but we have proved that *we may divide M in a maximal foliation such that any orbit is a union of leaves.* This foliation is maximal in the sense that there is no any other coarser foliation of M whose leaves are contained in the orbits (see Theorem 2 and Corollary 3).

Next, we will show that the leaves of $\mathcal{F}$ may be endowed with even more geometric structure. Indeed, we will construct a Lie groupoid structure over each leaf of $\mathcal{F}$.

For each $x \in M$, let us consider the minimal transitive groupoid $\overline{\Gamma}(\mathcal{F}(x))$ generated by the elements of $\overline{\mathcal{F}}(\epsilon(x))$. In fact, we may prove that

$$\overline{\Gamma}(\mathcal{F}(x)) = \sqcup_{\overline{g} \in \overline{\mathcal{F}}(\epsilon(x))} \overline{\mathcal{F}}\left(\epsilon\left(\alpha\left(\overline{g}\right)\right)\right), \tag{5}$$

i.e., $\overline{\Gamma}(\mathcal{F}(x))$ can be described as a disjoint union of fibres at the identities.

Theorem 2 *([5,19]). For each $x \in M$ there exists a transitive Lie subgroupoid $\overline{\Gamma}(\mathcal{F}(x))$ of Γ with base $\mathcal{F}(x)$.*

Thus, we have divided the manifold M into leaves $\mathcal{F}(x)$ which have a maximal structure of transitive Lie subgroupoids of Γ. The following result provides us an intuition about the maximality condition which satisfies the characteristic foliation and the base-characteristic foliation.

Corollary 3 *([5]). Let $\mathcal{H}$ be a foliation of M such that for each $x \in M$ there exists a transitive Lie subgroupoid $\Gamma(x)$ of Γ over the leaf $\mathcal{H}(x)$ contained in $\overline{\Gamma}$ whose family of leaves defines a foliation on Γ. Then, the base-characteristic foliation $\mathcal{F}$ is coarser than $\mathcal{H}$, i.e.,*

$$\mathcal{H}(x) \subseteq \mathcal{F}(x), \ \forall x \in M.$$

Furthermore, it satisfies that

$$\Gamma(x) \subseteq \overline{\Gamma}(\mathcal{F}(x)).$$

As a consequence we have that $\overline{\Gamma}$ is a transitive Lie subgroupoid of Γ if, and only if, $M = \mathcal{F}(x)$ and $\overline{\Gamma} = \overline{\Gamma}(\mathcal{F}(x))$ for some $x \in M$.

Thus, summarizing, for a fixed subgroupoid $\overline{\Gamma}$ of a Lie groupoid Γ we have available two canonical foliations, $\overline{\mathcal{F}}$ and $\mathcal{F}$ which endow to Γ of some kind of maximal differentiable structure. To study more properties of the characteristic distribution, we recommend [5].

Apart from Example 6, we may study several relevant applications of the characteristic distribution. In [5] we may find some of them. Here we are mainly interested in one of them, the so-called *material distributions*, which will be presented in what follows.

3. Material evolution of simple bodies

We will now present the notion of *simple material* mainly the references [33,37].

Definition 9. A *(deformable) body* is given by an oriented manifold $\mathcal{B}$ of dimension 3 which can be covered by just one chart. The points of $\mathcal{B}$ will be called *body points* or *material particles* and will be denoted by using capital letters $(X, Y, Z \in \mathcal{B})$. A *sub-body* of $\mathcal{B}$ is an open subset $\mathcal{U}$ of the manifold $\mathcal{B}$.

A *configuration* of the body $\mathcal{B}$ is an embedding $\phi : \mathcal{B} \to \mathbb{R}^3$, and an *infinitesimal configuration at a particle* X is given by the 1-jet $j^1_{X,\phi(X)}\phi$ where ϕ is a configuration of $\mathcal{B}$. The image $\phi(\mathcal{B})$ is called the *region occupied by the body* $\mathcal{B}$ *in the configuration* ϕ. The points on the Euclidean space $\mathbb{R}^3$ will be called *spatial points* and will be denoted by lower case letters $(x, y, z \in \mathbb{R}^3)$.

We will now fix a configuration, denoted by ϕ_0, called *reference configuration*. The image will be denoted by $\mathcal{B}_0 = \phi_0(\mathcal{B})$. Coordinates in the reference configuration will be denoted by X^I, while any other coordinates will be denoted by x^i.

Notice that, in [33] a body is defined simply as a three-dimensional manifold. Nevertheless, without loss of generality, we will adopt the above definition of body 9 which is used in [37]. On the other hand, in [25] a body is simple defined as an open subset of $\mathbb{R}^3$, so the body is identified with its image in $\mathbb{R}^3$ via a reference configuration.

Any change of configurations $\kappa = \phi_1 \circ \phi_0^{-1}$ or, equivalently a diffeomorphism from $\mathcal{B}_0$ to any other open subset $\mathcal{B}_1$ of $\mathbb{R}^3$, is called a *deformation*. Analogously, an *infinitesimal deformation at* $\phi_0(X)$ is given by a 1-jet $j^1_{\phi_0(X),\phi(X)}\kappa$, where κ is a deformation.

The change of the body in time will be modelized over the *body-time manifold*, which is defined as the fibre bundle $\mathcal{C} = \mathbb{R} \times \mathcal{B}$ over $\mathbb{R}$. Then, a *history* is given by a fibre bundle embedding $\Phi : \mathcal{C} \to \mathbb{R} \times \mathbb{R}^3$ over the identity.

Notice that Φ can be seen as a differentiable family of configurations $\phi_t : \mathcal{B} \to \mathbb{R}^3$ such that

$$\phi_t(x) = pr_{\mathbb{R}^3} \circ \Phi(t, x), \ \forall t \in \mathbb{R}, \ \forall x \in \mathcal{B}, \tag{6}$$

where $pr_{\mathbb{R}^3} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ is the projection on $\mathbb{R}^3$. Thus, Φ represent the evolution of the body in time t in such a way that the configuration of $\mathcal{B}$ at time t is ϕ_t. Then, at each instant of time t, one may consider the infinitesimal configuration at time t, 1-jet $j^1_{X,\phi_t(X)}\phi_t$.

Of course, the intrinsic properties of the body will play an important role in continuum mechanics. One of the most characteristic contributions of the work of W. Noll [26], was the introduction of the *mechanical response*; a differentiable map characterizing the internal properties of the material. In the case of *elastic simple bodies*, or simply *simple bodies*, [37] we will assume that the mechanical response depends on a particle only on the *infinitesimal deformation* at the same particle. Then, following [33,13,16], we will assume that, for a fixed reference configuration ϕ_0, the constitutive response at each material particle X and at each instant of time t, the mechanical response may be characterized by one (or more) functions depending on the associated matrices F to the infinitesimal configurations $j^1_{X,\phi_t(X)}\phi_t$ at particle X and time t.

Definition 10. Let $\mathcal{B}$ be a simple body with $\mathcal{C} = \mathbb{R} \times \mathcal{B}$ as the associated body time manifold. Then, for a fixed reference configuration ϕ_0, the *mechanical response* will be a differentiable map,

$$W : \mathcal{C} \times Gl(3, \mathbb{R}) \to V,$$

where V is again a real vector space.

Generally, in continuum mechanics, the contact forces at a particle X at an instant t in a given configuration ϕ are characterized by a symmetric second-order tensor $T_{t,X,\phi}$ on $\mathbb{R}^3$, which is called the stress tensor. Then, the mechanical response is given by the following equation:

$$W(t, X, F) = T_{t,X,\phi},$$

where F is the 1-jet at $\phi_0(X)$ of $\phi \circ \phi_0^{-1}$. Namely, in general, V will be the space of *stress tensors* [23] although, in this paper, we will only be interested in the structure of vector space of V. Notice that, the definition of the mechanical response permits us to compare material responses at different particles at different instants of time. Relevant examples are given by the volumetric growth and remodelling of biological tissues, such as bone and muscle [29].

Observe that, the construction of the mechanical response seems to be constrained to the fixed reference configuration. To clarify this dependence we have the *rule of change of reference configuration*.

Consider a different configuration ϕ_1 and W_1 its associated mechanical response. Then, it will be imposed that

$$W_1(t, X, F) = W(t, X, F \cdot C_{01}), \tag{7}$$

for all regular matrix F where C_{01} is the associated matrix to the 1-jet at $\phi_0(X)$ of $\phi_1 \circ \phi_0^{-1}$. Equivalently,

$$W(t, X, F_0) = W_1(t, X, F_1), \tag{8}$$

where F_i, $i = 0, 1$, is the associated matrix to the 1-jet at $\phi_i(X)$ of $\phi \circ \phi_i^{-1}$ with ϕ a configuration. Therefore, Eq. (7) permits us to define W over the space of (local) histories which is *independent on the chosen reference configuration*. In particular, for any history $\Phi = \phi_t$, we will define

$$W(t, X, \Phi) = W\left(t, j^1_{X,x}\phi_t\right) = W(t, X, F_t), \tag{9}$$

where F_t is the associated matrix to the 1-jet $j^1_{\phi_0(X),x}\left(\phi_t \circ \phi_0^{-1}\right)$ at $\phi_0(X)$.

Observe that, for all t the manifold $\{t\} \times \mathcal{B}$ inherits the structure of simple body by restricting the mechanical response W to the history of deformations at the same instant t (see [5]), i.e.,

$$W_t : \{t\} \times \mathcal{B} \times Gl(3, \mathbb{R}) \to V.$$

This body will be called *state t of the body* $\mathcal{B}$. Thus, we may think about W as a differentiable curve of mechanical responses, each one of these over the corresponding state of the body. As long as it invites no confusion, we will refer to the simple body $\{0\} \times \mathcal{B}$ as the material body $\mathcal{B}$.

On the other hand, it is also important to say that the mechanical response defines a structure of material evolution on any sub-body $\mathcal{U}$ of the body $\mathcal{B}$ by restriction. Nevertheless, we will need to relax the definition of "*material evolution*" to permit variation of material submanifolds a long time.

Definition 11. An *evolution material* for a *submanifold* (or *body-time generalized sub-body*) of $\mathcal{B}$ is a submanifold $\mathcal{M}$ of $\mathcal{C}$.

Thus, let us consider an instant t. Then, we have that the *state t of the material submanifold* is

$$(\{t\} \times \mathcal{B}) \cap \mathcal{M} = \{t\} \times \mathcal{M}_t,$$

for a submanifold $\mathcal{M}_t$ of $\mathcal{B}$. Hence, varying t, the model permits variations in the "*shape*" of $\mathcal{M}_t$. Notice that, because of this variation, in general, we will state that $\mathcal{M}$ is a evolution material for $\mathcal{M}_t$, for each t.

Maybe the cornerstone of the thesis of W. Noll [26] is the use of the so-called *material isomorphisms*. This notion arises from the need to respond the following question: when are two material points made of the same material? Obviously, we could say that two points are made of the same material if the constitutive response is exactly the same at both points. However, this is not enough general to answer the question.

Definition 12. Let $\mathcal{C}$ be a body-time manifold. Two pairs $(t, X), (s, Y) \in \mathcal{C}$ are said to be *materially isomorphic* if there exists a local diffeomorphism ψ from an open neighbourhood $\mathcal{U} \subseteq \mathcal{B}$ of X to an open neighbourhood $\mathcal{V} \subseteq \mathcal{B}$ of Y such that $\psi(X) = Y$ and

$$W(t, X, F \cdot P) = W(s, Y, F), \tag{10}$$

for all infinitesimal deformation F where P is given by the Jacobian matrix of $\phi_0 \circ \psi \circ \phi_0^{-1}$ at $\phi_0(X)$. The triples given by $\left(t, s, j^1_{X,Y}\psi\right)$, with ψ satisfying Eq. (10) are called *time-material isomorphisms* (or *material isomorphisms* if there is no danger of confusion) from (t, X) to (s, Y). A material isomorphism from (t, X) to itself is called a *time-material symmetry* or *material symmetry*. We will denote by $G(t, X)$ to the group of all material symmetries at (t, X).

Thus, the notion of being "*material isomorphic*" is the mathematical formulation of the intuitive idea of being made of the same of the material, in such a way that two material points (at maybe different instants) are made of the same material if, and only if, they are materially isomorphic.

Proposition 4. *Let $\mathcal{C}$ be a body-time manifold. Two body pairs (t, X) and (s, Y) are materially isomorphic if, and only if, there exist two (local) configurations ϕ_1 and ϕ_2 such that*

$$W_1(t, X, F) = W_2(s, Y, F), \ \forall F, \tag{11}$$

where W_i is the mechanical response associated to ϕ_i for $i = 1, 2$.

Proof. Consider $j^1_{X,Y}\psi$ a material isomorphism from (t, X) to (s, Y). We only have to choose $\phi_1 = \phi_0$ and

$$\phi_2 = \phi_1 \circ \psi \quad \square$$

So, this result shows the reason why the mathematical notion of material isomorphism answers to the question posed at the beginning of the section 3. In particular, two points are materially isomorphic if their constitutive properties are equal (up to the choice of the reference configuration). Roughly speaking, we may turn neighbourhoods of a material particle X at an instant t into neighbourhoods of the material particle Y at the other instant s, such that the *stress required to effect any given deformation* of those neighbourhoods is the same for each.

Remark 2. It is important to note that, in [33], the authors define the notion of *materially isomorphic* by using equation (11), next to an added condition referred to the *mass density*.

A non-negative scalar measure, $\mathfrak{m}$ defined on the body manifold $\mathcal{B}$ is called the *mass distribution* of the body. Then, $\mathfrak{m}$ induces a measure over each image $\phi(\mathcal{B})$ of $\mathcal{B}$ via a configuration ϕ denoted by $\mathfrak{m}_\phi$ which is assumed to be absolutely continuous with respect to the Lebesgue measure in Euclidean space $\mathbb{R}^3$. Therefore, by using the Radon-Nikodym theorem, we may construct a density ρ_ϕ, called *mass density associated to* ϕ. In fact, the relation between the mass and the density is the following,

$$\mathfrak{m}(\mathcal{P}) = \int_{\phi(\mathcal{P})} \rho_\phi \, dv,$$

for each every measurable subset $\mathcal{P}$ of $\mathcal{B}$. Here, the integral is defined in terms of Lebesgue measure in Euclidean space.

Finally, let us consider a sequence of suitable measurable subsets $\mathcal{P}_k$ having only the particle X in common that,

$$\lim_{k\to\infty} \mathfrak{v}(\mathcal{P}_k) = 0,$$

where $\mathfrak{v}$ defines the *volume map* respect to Lebesgue measure. Then, we define

$$\rho_\phi(X) = \lim_{k\to\infty} \frac{\mathfrak{m}_\phi(\mathcal{P}_k)}{\mathfrak{v}(\mathcal{P}_k)}$$

Thus, in [33], it is imposed that two material particles X and Y are materially isomorphic if, and only if, it satisfies Eq. (11) and

$$\rho_{\phi_1} = \rho_{\phi_2} = \text{constant}.$$

Namely, roughly speaking, material isomorphisms preserves the stress and the density. However, for the notion of material isomorphism, we will follow the philosophy of the book [37], where the authors state that, in static problems, since the inertia vanishes, this condition serves no purpose. ◇

4. Evolution material geometry

Let $\mathcal{V}$ be the vertical subbundle associated to the body-time manifold $\mathcal{C}$. Namely, $\mathcal{V}$ is the subbundle of the tangent bundle $T\mathcal{C}$ whose fibres are given by

$$\mathcal{V}_{(t,X)} = \{0\} \times T_X\mathcal{B},$$

for all $(t, X) \in \mathcal{C}$. Then, we may consider the associated frame groupoid $\Phi(\mathcal{V}) \rightrightarrows \mathcal{C}$ introduced in Example 4. In fact, the elements of $\Phi(\mathcal{V})$ are the linear isomorphisms between fibres of $\mathcal{V}$. We may represent the elements of $\Phi(\mathcal{V})$ in two different ways:

1.- Let $\Phi : \mathcal{C} \to \mathcal{C}$ be a (local) embedding of fibre bundles. Then, the triple $\left((t, X), \Phi(t, X), j^1_{X,\phi_t(X)}\phi_t\right)$, with $\phi_t : \mathcal{B} \to \mathcal{B}$ the associated family of configurations given in Eq. (6), characterizes a linear isomorphism

$$L_{(t,X),\Phi(t,X)} : \mathcal{V}_{(t,X)} \to \mathcal{V}_{\Phi(t,X)}$$

Reciprocally, any linear isomorphism between fibres of $\mathcal{V}$ may be represented as a triple generated by a (local) embedding of fibre bundles $\Phi : \mathcal{C} \to \mathcal{C}$.

2.- Another, less intuitive but easier, way to represent an element of $\Phi(\mathcal{V})$ is a triple $\left(t, s, j^1_{X,Y}\phi\right)$ with $s, t \in \mathbb{R}$, $X \in \mathcal{B}$ and ϕ a local diffeomorphism from $\mathcal{B}$ to $\mathcal{B}$ such that $\phi(X) = Y$.

To define the structure of differentiable manifold on $\Phi(\mathcal{V})$, we will use the second representation of the elements of $\Phi(\mathcal{V})$. So, the local coordinates of $\Phi(\mathcal{V})$ (see example (5)) are given by

$$\Phi(\mathcal{V}_\mathcal{U}) : \left(t, s, x^i, y^j, y^j_i\right), \tag{12}$$

where, for each $\left(t, s, j^1_{X,Y}\phi\right) \in \Phi(\mathcal{V}_\mathcal{U})$

- $t\left(t,s,j^1_{X,Y}\phi\right)=t$
- $s\left(t,s,j^1_{X,Y}\phi\right)=s$
- $x^i\left(t,s,j^1_{X,Y}\phi\right)=x^i(X)$
- $y^j\left(t,s,j^1_{X,Y}\phi\right)=y^j(Y)$
- $y^j_i\left(t,s,j^1_{X,Y}\phi\right)=\dfrac{\partial\left(y^j\circ\phi\right)}{\partial x^i_{|X}}$

where $\left(x^i\right)$ and $\left(y^j\right)$ are local charts defined on the open subsets of $\mathcal{B}$, $\mathcal{U}$ and $\mathcal{W}$, respectively, and $\Phi\left(\mathcal{V}_{\mathcal{U},\mathcal{W}}\right)$ is given by the triples $\left(t,s,j^1_{X,Y}\phi\right)$ such that $X\in\mathcal{U}$ and $Y\in\mathcal{W}$.

The importance of $\Phi(\mathcal{V})$ lies in the fact that all the material isomorphisms (see definition (10)) are elements of $\Phi(\mathcal{V})$. On the other hand, the mechanical response may be defined on this groupoid. Indeed, by using Eq. (9), we may define W on the frame groupoid of $\mathcal{V}$,

$$W:\Phi(\mathcal{V})\to V,$$

as follows,

$$W\left(t,s,j^1_{X,Y}\phi\right)=W(t,X,\Phi),$$

such that

$$\Phi(s,Y)=(s,\phi_0\circ\phi(Y)),\ \forall(s,Y)\in\mathcal{C},$$

where ϕ_0 is the reference configuration. Notice that, in consequence, W does not depend on the final point, i.e., for all $(t,X),(s,Y),(r,Z)\in\mathcal{C}$

$$W\left(t,s,j^1_{X,Y}\phi\right)=W\left(t,r,j^1_{X,Z}\left(\phi_0^{-1}\circ\tau_{Z-Y}\circ\phi_0\circ\phi\right)\right),\tag{13}$$

for all $\left(t,s,j^1_{X,Y}\phi\right)\in\Phi(\mathcal{V})$ where τ_v is the translation map on $\mathbb{R}^3$ by the vector v. This point of view will be useful for our purpose.

On the other hand, we may define the *material groupoid* of a body-time manifold with mechanical response W as the largest subgroupoid $\Omega(\mathcal{C})\rightrightarrows\mathcal{C}$ of $\Phi(\mathcal{V})$ such that leaves W invariant. More explicitly, an element of $\Phi(\mathcal{V})$ $\left(t,s,j^1_{X,Y}\phi\right)$ is in the material groupoid, if and only, if

$$W\left(t,r,j^1_{X,Z}(\psi\cdot\phi)\right)=W\left(s,r,j^1_{Y,Z}\psi\right),$$

for all $\left(s,r,j^1_{Y,Z}\psi\right)\in\Phi(\mathcal{V})$. In other words, $\Omega(\mathcal{C})$ is the space of all (time-)material isomorphisms (see Definition 12). This groupoid was first presented in [14].

The isotropy group at each $(t,X)\in\mathcal{C}$ will be denoted by $G(t,X)$ and its elements are the material symmetries at (t,X). Observe that, as in the spatial case [5,19], the resulting groupoid does not have to be a Lie subgroupoid.

We will also define the (X,Y)-*material groupoid* $\Omega_{X,Y}(\mathbb{R})$ as the set of all material isomorphisms from the particle X to the particle Y varying the time variable. Observe that, when $X=Y$, the (X,X)-material groupoid $\Omega_{X,X}(\mathbb{R})$ is a subgroupoid of the material groupoid $\Omega(\mathcal{C})$. For each material point X, $\Omega_{X,X}(\mathbb{R})$ is called X-*material groupoid* and denoted by $\Omega_X(\mathbb{R})$.

On the other hand, $\Omega_X(\mathbb{R})$ may be considered as a subgroupoid of $(\mathbb{R}\times\mathbb{R})\times\Pi^1(\mathcal{B},\mathcal{B})^X_X$ on $\mathbb{R}$, where we are identifying $\mathbb{R}$ with $\mathbb{R}\times\{X\}$. Furthermore, the structure of Lie groupoid of $(\mathbb{R}\times\mathbb{R})\times\Pi^1(\mathcal{B},\mathcal{B})^X_X$ is given by

$$\left(s,t,j^1_{X,X}\phi\right)\cdot\left(r,s,j^1_{X,X}\psi\right)=\left(r,t,j^1_{X,X}(\phi\circ\psi)\right),$$

for all $\left(s,t,j^1_{X,X}\phi\right),\left(r,s,j^1_{X,X}\psi\right)\in\mathbb{R}\times\mathbb{R}\times\Pi^1(\mathcal{B},\mathcal{B})^X_X$ (see Example 3). Again, we will use both interpretations of $\Omega_X(\mathbb{R})$ along the paper.

Proposition 5. *Let $\Omega(\mathcal{C})$ be the material groupoid. If $\Omega(\mathcal{C})$ is a Lie subgroupoid of $\Phi(\mathcal{V})$, then for all material point X we have that $\Omega_X(\mathbb{R})$ is a Lie subgroupoid of $\Phi(\mathcal{V})$.*

Proof. Assume that $\Omega(\mathcal{C})$ is a Lie subgroupoid of $\Phi(\mathcal{V})$. Let us consider the following submersions

$$\pi_1 : \Omega(\mathcal{C}) \to \mathcal{B} \times \mathcal{B},$$

given by

$$\pi_1\left(t, s, j^1_{X,Y}\phi\right) = (X, Y),$$

for all $\left(t, s, j^1_{X,Y}\phi\right) \in \Omega(\mathcal{C})$. Then

$$\Omega_X(\mathbb{R}) = \pi_1^{-1}(X, X). \quad \square$$

So, the imposition of *"being a Lie groupoid"* is stronger over the material groupoid than over the X-material groupoids. Notice the material groupoid $\Omega(\mathcal{C})$ encompasses the whole evolution of the body $\mathcal{B}$ and the X-material groupoid $\Omega_X(\mathbb{R})$ codifies the evolution of the particle X.

Therefore, as a summary, we have some canonical subgroupoids ($\Omega(\mathcal{C})$ and $\Omega_X(\mathbb{R})$) of a particular Lie groupoid ($\Phi(\mathcal{V})$), i.e., we are facing a situation which fits in the framework of the construction of the characteristic distribution (see section 2).

In [17,22], the authors provide a specific representation of the corresponding characteristic distributions. In fact, the associated characteristic distribution $A\Omega(\mathcal{C})^T$ to the material groupoid, which will be called *material distribution* of the body-time manifold $\mathcal{C}$ is generated by the (left-invariant) vector fields Θ on $\Phi(\mathcal{V})$ which are in the kernel of TW, i.e.,

$$TW(\Theta) = 0. \tag{14}$$

Namely, let Θ be a left-invariant vector field on $\Phi(\mathcal{V})$. Then,

$$\Theta\left(t, s, x^i, y^j, y^i_j\right) = \lambda\frac{\partial}{\partial t} + \Theta^i\frac{\partial}{\partial x^i} + y^i_l\Theta^l_j\frac{\partial}{\partial y^i_j} \tag{15}$$

respect to a local system of coordinates $\left(t, s, x^i, y^j, y^i_j\right)$ on $\Phi\left(\mathcal{V}_{\mathcal{U},\mathcal{V}}\right)$ with $\mathcal{U}$ and $\mathcal{V}$ two open subsets of $\mathcal{B}$ and $\Phi\left(\mathcal{V}_{\mathcal{U},\mathcal{V}}\right)$ is given by the triples $\left(t, s, j^1_{X,Y}\phi\right)$ in $\Phi(\mathcal{V})$ such that $X \in \mathcal{U}$ and $Y \in \mathcal{V}$. Then, Θ is an admissible vector field for the pair $(\Phi(\mathcal{V}), \Omega(\mathcal{C}))$ if, and only if, the following equations hold,

$$\lambda\frac{\partial W}{\partial t} + \Theta^i\frac{\partial W}{\partial x^i} + y^i_l\Theta^l_j\frac{\partial W}{\partial y^i_j} = 0. \tag{16}$$

Notice that, here λ, Θ^i and Θ^i_j are function depending on t and X. Eq. (16) is the so-called *evolution equation*, which is a tool to characterize *remodelling* and *aging* [22].

Thus, to construct the material distribution, we have to solve the evolution equation (16). The base-characteristic distribution $A\Omega(\mathcal{C})^\sharp$ will be called *body-material distribution*.

The foliations associated to the material distribution and the body-material distribution is called *material foliation* and *body-material foliation* and they will be denoted by $\overline{\mathcal{F}}$ and $\mathcal{F}$, respectively.

The characteristic distribution associated to the X-material groupoid $A\Omega_X(\mathbb{R})^T$ is called *X-material distribution*. Analogously, $A\Omega_X(\mathbb{R})^T$ is generated by the (left-invariant) vector fields on $(\mathbb{R} \times \mathbb{R}) \times \Pi^1(\mathcal{B}, \mathcal{B})^X_X$ which are in the kernel of TW_X, where W_X is given by the restriction of W to $\mathbb{R} \times \mathbb{R} \times \Pi^1(\mathcal{B}, \mathcal{B})^X_X$,

$$W_X : \mathbb{R} \times \mathbb{R} \times \Pi^1(\mathcal{B}, \mathcal{B})^X_X \to V$$

In other words, the X-material distribution of $\mathcal{C}$ is generated by the left-invariant vector fields Θ on $\mathbb{R} \times \mathbb{R} \times \Pi^1(\mathcal{B}, \mathcal{B})^X_X$ such that

$$TW_X(\Theta) = 0. \tag{17}$$

Let Θ be a left-invariant vector field on $\mathbb{R} \times \mathbb{R} \times \Pi^1(\mathcal{B}, \mathcal{B})^X_X$. Then,

$$\Theta\left(t, s, y^i_j\right) = \lambda\frac{\partial}{\partial t} + y^i_l\Theta^l_j\frac{\partial}{\partial y^i_j}, \tag{18}$$

respect to a local system of coordinates $\left(t, s, y^i_j\right)$ on $\mathbb{R} \times \mathbb{R} \times \Pi^1(\mathcal{U}, \mathcal{U})^X_X$ with $\mathcal{U}$ an open subset of $\mathcal{B}$ with $X \in \mathcal{U}$. Then, Θ is an admissible vector field for the pair $\left(\Phi(\mathcal{V}), \Omega_X(\mathbb{R})\right)$ if, and only if, the following equations hold,

$$\lambda \frac{\partial W_X}{\partial t} + y^i_l \Theta^l_j \frac{\partial W_X}{\partial y^i_j} = 0 \tag{19}$$

Observe that, here λ and Θ^i_j are function depending on t. This equation is called the *evolution equation at* X [17].

The base-characteristic distribution $A\Omega_X(\mathbb{R})^\sharp$ (see Theorem 1) will be called *X-body-material distribution*. The foliations associated to the X-material distribution and the X-body-material distribution will be called *X-material foliation* and *X-body-material foliation*, and they will be denoted by $\overline{\mathcal{F}}_X$, respectively. It is important not to confuse $\overline{\mathcal{F}}_X$ (resp. $\mathcal{F}_X$), the X-material foliation (resp. X-body-material foliation), with $\overline{\mathcal{F}}(\epsilon(X))$ (resp. $\mathcal{F}(X)$), the leaf at $\epsilon(X)$ (resp. the leaf at X) of the foliation $\overline{\mathcal{F}}$ (res. $\mathcal{F}$).

5. Morphogenesis

As opposed to the uniformity in the spatial case [5], new material properties may arise during the evolution of the body. In particular, the temporal counterpart of uniformity is a specific case of evolution of the material called *remodelling*.

Definition 13. Let $\mathcal{C}$ be a body-time manifold. A material particle $X \in \mathcal{B}$ is presenting a *remodelling* when it is connected with all the instants by a material isomorphism, i.e., all the points at $\mathbb{R} \times \{X\}$ are connected by material isomorphisms. $\mathcal{C}$ is presenting a *remodelling* when all the material points are presenting a remodelling. *Remodelling growth* and *remodelling resorption* are given by a remodelling with volume increase or volume decrease of the material body $\mathcal{B}$.

Intuitively, a material evolution presents a remodelling when the constitutive properties of the material do not change with the time. This kind of evolution may be found in biological tissues [29]. Wolff's law of trabecular architecture of bones (see for instance [35]) is a relevant example. Here, trabeculae are assumed to change their orientation following the principal direction of stress. It is important to note that the fact of that the material body remains materially isomorphic with the time does not preclude the possibility of adding (remodelling growth) or removing (remodelling resorption) material, as long as the material added is *of the same type*. It is easy to realize that a particle X *is presenting a remodelling if, and only if, the X-material groupoid is transitive* [17]. On the other hand, $\mathcal{C}$ *is presenting a remodelling if, and only if, the X-material groupoid is transitive for any material point* X. Finally, $\mathcal{C}$ *is presenting a remodelling if, and only if, the material groupoid is transitive* [22].

Definition 14. Let $\mathcal{C}$ be a body-time manifold. A material particle $X \in \mathcal{B}$ is presenting a *aging* when it is not presenting a remodelling, i.e., not all the instants are connected by a material isomorphism. $\mathcal{C}$ is a *process of aging* if it is not a process of remodelling.

Clearly, if the material response is not preserved along the time via material isomorphism, the constitutive properties are changing with the time. Although it is something natural, a proper definition of *smooth aging* was not obtained until now [17,22].

Proposition 6 *([17]). Let* $\mathcal{C}$ *be a body-time manifold. A material particle* $X \in \mathcal{B}$ *is presenting an aging if, and only if, the X-material groupoid* $\Omega_X(\mathbb{R})$ *is not transitive.* $\mathcal{C}$ *is presenting an aging if, and only if, for some material point* X*, the X-material groupoid* $\Omega_X(\mathbb{R})$ *is not transitive.*

In [17,22], the authors use the corresponding material distributions and foliations to presents several results characterizing the different types of remodelling and aging. Here, however, we are interested in a particular case of evolution: *morphogenesis*.

We recall the notion of a normal group. Let G be a group. A subgroup N of G is said to be *normal*, and denoted by $N \trianglelefteq G$, if it is invariant under conjugation, i.e.,

$$N = g \cdot N \cdot g^{-1}, \tag{20}$$

for all $g \in G$. Let us consider a general subgroup H of G. Then, the *normalizer* of H in G, denoted by $\mathcal{N}(H)$, is defined as the family of elements of G in such a way that H is invariant under their conjugations, i.e.,

$$\mathcal{N}(H) := \{g \in G \ : \ H = g \cdot H \cdot g^{-1}\} \tag{21}$$

In fact, $\mathcal{N}(H)$ is the largest subgroup of G in which H is a normal subgroup.

In order to work deal with the concept of *morphogenesis*, we will need to extend this notion to groupoids. This generalization may be found in [2] to study uniformity and homogeneity in *functional graded materials* (FGM). Here, we will develop a previous mathematical study of this extension, which is necessary for our research.

Definition 15. Let $\Gamma \rightrightarrows M$ be a groupoid. A subgroupoid $\overline{\Gamma} \rightrightarrows N$ of Γ is said to be *normal* in Γ if it is invariant under conjugation, i.e.,

$$g \cdot h \cdot g^{-1} \in \overline{\Gamma},$$

for all $g \in \Gamma$ and $h \in \overline{\Gamma}$ such that $\beta(h) = \alpha(g) = \alpha(h)$.

Notice that, in the case of groupoid, an equation like Eq. (20) does not make sense because, in general, we cannot compose an element of Γ with all the elements at $\overline{\Gamma}$. In general, one could think that the imposition of being a normal subgroupoid may be reduced to the isotropy groups in such a way that *a subgroupoid of a groupoid is normal if, and only if, all the isotropy groups are normal in the corresponding isotropy groups of the groupoid*. Nevertheless, in general, condition of being a normal subgroupoid is more restrictive than the property of that all the isotropy groups are normal subgroups of the corresponding isotropy groups.

An easy counterexample is the following: consider the groupoid $\Gamma \rightrightarrows M$ such that $\Gamma = M \times M \times Gl(3, \mathbb{R})$ and $M := \{x, y\}$ (see Example 3). Here, we may consider the subgroupoid $\overline{\Gamma} \rightrightarrows M$ characterized by:

- $\overline{\Gamma}_x^x = \{x\} \times \{x\} \times Gl(3, \mathbb{R})$
- $\overline{\Gamma}_y^y = \{y\} \times \{y\} \times \{Id\}$
- $\overline{\Gamma}_x^y = \emptyset$

Then, obviously, all the isotropy groups of $\overline{\Gamma}$ are normal in the corresponding isotropy groups of Γ. However, in general, an element F of $\overline{\Gamma}_x^x = \{x\} \times \{x\} \times Gl(3, \mathbb{R})$ and an element H of $\Gamma_x^y = \{y\} \times \{y\} \times Gl(3, \mathbb{R})$ we do not have

$$H \cdot F \cdot H^{-1} \in \overline{\Gamma}_y^y = \{y\} \times \{y\} \times \{Id\}.$$

So, $\overline{\Gamma}$ is not a normal subgroupoid of Γ.

Proposition 7. *Let $\Gamma \rightrightarrows M$ be a groupoid. A transitive subgroupoid $\overline{\Gamma} \rightrightarrows N$ of Γ is normal in Γ if, and only if, all the isotropy groups are normal in the corresponding isotropy groups of Γ.*

Proof. Assume that all isotropy groups of $\overline{\Gamma}$ are normal in the corresponding isotropy groups of Γ. Then, for all $x \in M$ we have

$$\overline{\Gamma}_x^x \trianglelefteq \Gamma_x^x, \ \forall x \in N.$$

Let $g \in \Gamma$ and $h \in \overline{\Gamma}$ such that, $\beta(h) = \alpha(g) = \alpha(h)$. Then, by transitivity, consider $l \in \overline{\Gamma}$ such that $\alpha(l) = \alpha(h) = \beta(h) = \alpha(g)$ y $\beta(l) = \beta(g)$. Then,

$$g \cdot l^{-1} \in \Gamma_{\beta(g)}^{\beta(g)} \quad , \quad l \cdot h \cdot l^{-1} \in \overline{\Gamma}_{\beta(g)}^{\beta(g)}$$

Therefore,

$$g \cdot h \cdot g^{-1} = \left(g \cdot l^{-1}\right) \cdot \left(l \cdot h \cdot l^{-1}\right) \cdot \left(g \cdot l^{-1}\right)^{-1} \in \overline{\Gamma}_{\beta(g)}^{\beta(g)} \quad \square$$

So, we have a wide family of subgroupoids in which the imposition of being normal may be reduced to the corresponding imposition on the isotropy groups.

Definition 16. Let $\Gamma \rightrightarrows M$ be a groupoid and $\overline{\Gamma} \rightrightarrows N$ be a subgroupoid of Γ. The *normalizoid of $\overline{\Gamma}$ in Γ* is given by

$$\mathcal{N}\left(\overline{\Gamma}\right) := \{g \in \Gamma \ / g \cdot h \cdot g^{-1} \in \overline{\Gamma}, \ \forall h \in \overline{\Gamma}, \ \beta(h) = \alpha(g) = \alpha(h)\} \tag{22}$$

Proposition 8. *Let $\Gamma \rightrightarrows M$ be a groupoid and $\overline{\Gamma} \rightrightarrows N$ be a subgroupoid of Γ. The normalizoid $\mathcal{N}\left(\overline{\Gamma}\right)$ of $\overline{\Gamma}$ in Γ has a structure of subgroupoid of Γ over N. In fact, $\mathcal{N}\left(\overline{\Gamma}\right)$ is the largest subgroupoid of Γ satisfying that $\overline{\Gamma}$ is a normal subgroupoid of $\mathcal{N}\left(\overline{\Gamma}\right)$.*

Proof. We only have to check that the composition is closed in $\mathcal{N}\left(\overline{\Gamma}\right)$. $\mathcal{N}\left(\overline{\Gamma}\right)$ is the largest subgroupoid of Γ satisfying that $\overline{\Gamma}$ is a normal subgroupoid of $\mathcal{N}\left(\overline{\Gamma}\right)$ by construction. $\square$

Corollary 9. *Let $\Gamma \rightrightarrows M$ be a groupoid and $\overline{\Gamma} \rightrightarrows N$ be a subgroupoid of Γ. The isotropy groups of the normalizoid of $\overline{\Gamma}$ in Γ are the normalizers of the isotropy groups of $\overline{\Gamma}$ in the corresponding isotropy groups of Γ, i.e.,*

$$\mathcal{N}\left(\overline{\Gamma}\right)_x^x = \mathcal{N}\left(\overline{\Gamma}_x^x\right),$$

for all $x \in N$.

Proof.

$$\begin{aligned}\mathcal{N}\left(\overline{\Gamma}\right)_x^x &= \{g \in \Gamma \,/\, g \cdot h \cdot g^{-1} \in \overline{\Gamma},\ \forall h \in \overline{\Gamma},\ \beta(h) = \alpha(g) = \alpha(h) = x = \beta(g)\}\\ &= \{g \in \Gamma_x^x \,/\, g \cdot h \cdot g^{-1} \in \overline{\Gamma},\ \forall h \in \overline{\Gamma},\ \beta(h) = \alpha(g) = \alpha(h),\}\\ &= \{g \in \Gamma_x^x \,/\, g \cdot h \cdot g^{-1} \in \overline{\Gamma},\ \forall h \in \overline{\Gamma}_x^x,\}\\ &= \mathcal{N}\left(\overline{\Gamma}_x^x\right) \quad \square\end{aligned}$$

Of course, a deeper study of the normalizoid as an abstract structure in the theory of groupoid has a great interest from a mathematical point of view. Nevertheless, this could distract the reader from the main goal of this paper. So, we will leave the mathematical study of this notion here and, from now on, we will be focused on its application to materials.

The *extended material groupoid of* $\mathcal{C}$, denoted by $\mathcal{N}(\mathcal{C})$, is defined as the normalizoid of the material groupoid of $\mathcal{C}$, which is a subgroupoid of $\Phi(\mathcal{V})$, i.e.,

$$\mathcal{N}(\mathcal{C}) = \mathcal{N}(\Omega(\mathcal{C})).$$

Thus, more explicitly,

The elements of $\mathcal{N}(\mathcal{C})$ *are the triples* $\left(t, s, j^1_{X,Y}\psi\right)$*, with* ψ *a local automorphism on* $\mathcal{B}$*, such that,* $\left(s, s, j^1_{Y,Y}\left(\psi \circ \phi \circ \psi^{-1}\right)\right)$ *is a material isomorphism, for all material symmetry* $\left(t, t, j^1_{X,X}\phi\right)$ *on* (t, X).

Notice that, by Proposition 8, *all the material isomorphisms are contained in* $\mathcal{N}(\mathcal{C})$. In fact, $\mathcal{N}(\mathcal{C})$ is the largest subgroupoid of $\Phi(\mathcal{V})$ satisfying that $\Omega(\mathcal{C})$ is a normal subgroupoid of $\mathcal{N}(\mathcal{C})$.

Analogously we define the *X-extended material groupoid of* $\mathcal{C}$, denoted by $\mathcal{N}_X(\mathbb{R})$, as the normalizoid of the X-material groupoid of $\mathcal{C}$. Thus, in general, we have the following short exact sequences,

$$\Omega(\mathcal{C}) \leq \mathcal{N}(\mathcal{C}) \leq \Phi(\mathcal{V})$$
$$\Omega_X(\mathbb{R}) \leq \mathcal{N}_X(\mathbb{R}) \leq \mathbb{R} \times \mathbb{R} \times \Pi^1(\mathcal{B}, \mathcal{B})_X^X$$

An example of aging process may be the weakening of the stiffness of a mineralized bone due to hormonal deficiencies. However, the resulting bone could still be isotropic. Then, there has not been a qualitative change of the constitutive properties. This fact, leads to identify a particular kind of aging: *morphogenesis*.

Definition 17. Let $\mathcal{C}$ be a body-time manifold. A material point X is said to be undergone a *process of evolution without morphogenesis* when its symmetry groups at all the instants are conjugated. $\mathcal{C}$ is said to be undergone a *process of evolution without morphogenesis* when all the points are undergone process of evolution without morphogenesis.

Let us clarify the definition. Fix a material particle X of the body. Remember that, for each instant t, the symmetry group $G(t, X)$ is given by all the material isomorphisms from (t, X) to (t, X), i.e.,

$$G(t, X) = \Omega(\mathcal{C})_{(t,X)}^{(t,X)} = \Omega_X(\mathbb{R})_t^t$$

Then, X is presenting a process of evolution without morphogenesis iff for any two instant t and s, there exists a triple $\left(s, t, j^1_{X,X}\phi\right) \in \Phi(\mathcal{V})$ such that

$$G(t, X) = \left(s, t, j^1_{X,X}\phi\right) \cdot G(s, X) \cdot \left(t, s, j^1_{X,X}\phi^{-1}\right) \tag{23}$$

Suppose that $\mathcal{C}$ is undergone a remodelling process. Then, connecting via material isomorphisms, all the symmetry groups are conjugated. Hence, *remodelling is an example of process of evolution without morphogenesis*. In other words, as it is natural, any process of morphogenesis is a particular case of aging.

Nevertheless, not all the processes of evolution without morphogenesis are remodellings. In particular, we have a process of aging without morphogenesis when not all the symmetry groups may be conjugated by material isomorphisms, but for arbitrary elements of $\Phi(\mathcal{V})$.

Thus, a process of morphogenesis entails a *breakdown of symmetry*, a sudden change in the material symmetry type. A state of instability could provoke that the deviations from spherical symmetry of an embryo (in its spherical blastula stage) tend to grow reaching a new equilibrium in which the symmetries have changed (see [34]).

Many other king of morphogenesis may be happened in the realm of solids bodies. In [13] the author study how could change the *type of symmetry* from isotropy to transverse isotropy or orthotropy.

Proposition 10. *Let $\mathcal{C}$ be a body-time manifold. A material point X is undergone a process of evolution without morphogenesis if, and only if, the X-extended material groupoid $\mathcal{N}_X(\mathbb{R})$ is transitive. $\mathcal{C}$ is presenting a process of evolution without morphogenesis if, and only if, the extended material groupoid $\mathcal{N}(\mathcal{C})$ is transitive.*

Proof. Notice that, in particular, a material point X presents a process of evolution without morphogenesis if, and only if, for each two instants t and s, there exists an element $g = \left(s, t, j^1_{X,X}\phi\right)$ of $\Phi(\mathcal{V})$ from (s, X) to (t, X) such that

$$\Omega_X(\mathbb{R})^t_t = g \cdot \Omega_X(\mathbb{R})^s_s \cdot g^{-1}, \tag{24}$$

where $\Omega_X(\mathbb{R})^r_r$ is the isotropy group at (r, X) for all s. In other words, X presents a process of evolution without morphogenesis if, and only if, for each two instants t and s, there exists an element g of the X-extended material groupoid $\mathcal{N}_X(\mathbb{R})$ from (t, X) to (s, X) or, in other words, if and only if $\mathcal{N}_X(\mathbb{R})$ is transitive. The proof for $\mathcal{N}(\mathcal{C})$ is analogous. □

So, the normalizoid of the X-material groupoids works as a tool to study the morphogenesis of a particle, while the normalizoid of the material groupoid works to study the morphogenesis of the whole body, both via the property of transitivity. However, in general, it is not easy to check the transitivity of a groupoid. To solve this problem, we will use the associated characteristic distribution.

Now, we will consider the associated characteristic distribution $A\mathcal{N}(\mathcal{C})^T$ to the extended material groupoid of $\mathcal{C}$, which will be called *extended material distribution*. The base-characteristic distribution $A\mathcal{N}(\mathcal{C})^\sharp$ will be called *extended body-material distribution*.

The foliations associated to the extended material distribution and the extended body-material distribution will be called *extended material foliation* and *extended body-material foliation* and they will be denoted by $\mathcal{NF}$ and $\overline{\mathcal{NF}}$, respectively.

Then, as a consequence of Theorem 2 and Corollary 3, we have that,

Theorem 11. *For each $(X, t) \in \mathcal{C}$, there exists a transitive Lie subgroupoid $\mathcal{N}(X, t)$ of $\Phi(\mathcal{V})$ with base $\mathcal{NF}(X, t)$, which is totally contained in $\mathcal{N}(\mathcal{C})$. Furthermore, $\mathcal{NF}$ is the coarsest foliation of $\mathcal{C}$ satisfying this property.*

Notice that, over each leaf $\mathcal{NF}(X, t)$, there exists a *transitive Lie* subgroupoid $\mathcal{N}(X, t)$ contained in $\mathcal{N}(\mathcal{C})$. This fact is interpreted as each leaf is a differentiable process of morphogenesis. Thus, we would like to highlight that this mathematical result has a very intuitive meaning:

> *The body-time manifold $\mathcal{C}$ may be canonically divided, in a maximal way, into submanifolds (the foliation $\mathcal{NF}$) in such a way that all the leaves are differentiable processes of evolution without morphogenesis.*

As we commented, to study the morphogenesis is not an easy problem using unically the transitivity of the extended material groupoid. Then, following result present a "*computable*" way to study morphogenesis via Theorem 11.

Theorem 12. *Let $\mathcal{C}$ be a body-time manifold and $\mathcal{N}(\mathcal{C})$ its extended material groupoid. The extended material distribution $A\mathcal{N}(\mathcal{C})^T$ is pointwise generated by the left-invariant vector fields Θ on $\Phi(\mathcal{V})$ satisfying that*

$$TW([\Theta, \Lambda]) = 0, \tag{25}$$

for all left invariant admissible vector field Λ for the pair $(\Phi(\mathcal{V}), \Omega(\mathcal{C}))$ which is tangent to the α-fibres.

Proof. Let $\Theta \in \mathfrak{X}_{loc}(\Phi(\mathcal{V}))$ be a (local) left-invariant vector field on $\Phi(\mathcal{V})$ whose (local) flow is denoted by φ^Θ_t. Let us also denote by $\Theta^\sharp \in \mathfrak{X}_{loc}(\mathcal{C})$ the (local) the projection of Θ on the base-characteristic distribution of $\Phi(\mathcal{V})$, i.e.,

$$\Theta^\sharp = T\alpha \circ \Theta \circ \epsilon$$

The local flow of $\Theta^\sharp$ will be denoted by $\varphi^{\Theta^\sharp}_t$. Notice that, we have $\varphi^{\Theta^\sharp}_t = \alpha \circ \varphi^\Theta_t \circ \epsilon$.

Then, by definition, Θ is admissible for the pair $(\Phi(\mathcal{V}), \mathcal{N}(\mathcal{C}))$ if, and only if, φ^Θ_t is completely contained in $\mathcal{N}(\mathcal{C})$ at the identities. In other words,

$$W\left(F \cdot \varphi^\Theta_t(\epsilon(r, X)) \cdot h \cdot \varphi^\Theta_{-t}\left(\epsilon\left(\varphi^{\Theta^\sharp}_t(r, X)\right)\right)\right) = W(F), \tag{26}$$

for all $(r, X) \in \mathcal{C}$, $F \in \Phi(\mathcal{V})$ and $h \in \Omega(\mathcal{C})$ with $\alpha(F) = (r, X)$ and $\alpha(h) = \beta(h) = \varphi_t^{\Theta^\sharp}(r, X)$.

By simplicity, the identity at (r, X) in $\Phi(\mathcal{V})$, $\left(r, r, j^1_{X,X} Id_{T_X\mathcal{B}}\right)$ with $Id_{T_X\mathcal{B}}$ the identity map on $T_X\mathcal{B}$, is denoted by $\epsilon(r, X)$.

Let us now consider a left invariant vector field Λ on $\Phi(\mathcal{V})$ tangent to the α-fibres whose (local) flow is denoted by φ_t^Λ. Then, it satisfies

$$\begin{aligned} & W\left(F \cdot \varphi_t^\Theta(\epsilon(r, X)) \cdot \varphi_s^\Lambda\left(\epsilon\left(\varphi_t^{\Theta^\sharp}(r, X)\right)\right) \cdot \varphi_{-t}^\Theta\left(\epsilon\left(\varphi_t^{\Theta^\sharp}(r, X)\right)\right)\right) \\ & = W\left(\varphi_{-t}^\Theta\left(\varphi_s^\Lambda\left(\varphi_t^\Theta(F)\right)\right)\right) \\ & = W(F) \end{aligned}$$

Notice that, since it is tangent to the α-fibres, φ_t^Λ have the same projection under either β or α (so, the above composition makes sense). Then, differentiating with respect to the variable s at 0, we get

$$T_F W\left(\varphi_{-t_*}^\Theta \Lambda(F)\right) = 0, \tag{27}$$

where $\varphi_{-t_*}^\Theta \Lambda$ is the pushforward of Λ by ϕ_{-t}^Θ. Therefore, differentiating with respect to the variable t at 0, we have that

$$T_F W([\Theta, \Lambda](F)) = 0. \tag{28}$$

Conversely, assume that Eq. (28) is satisfied for all left invariant admissible vector field Λ, tangent to the α fibres, for the material distribution $A\Omega(\mathcal{C})^T$. Using the same notation for the (local) flows, notice that, in general

$$\begin{aligned} 0 &= \frac{\partial}{\partial t_{|0}}\left(\frac{\partial}{\partial s_{|0}}\left(W\left(\varphi_{-t}^\Theta\left(\varphi_s^\Lambda\left(\varphi_t^\Theta(F)\right)\right)\right)\right)\right) \\ &= -\frac{\partial}{\partial t_{|0}}\left(\frac{\partial}{\partial s_{|0}}\left(W\left(\varphi_{-t}^\Lambda\left(\varphi_s^\Theta\left(\varphi_t^\Lambda(F)\right)\right)\right)\right)\right) \\ &= -\frac{\partial}{\partial t_{|0}}\left(\frac{\partial}{\partial s_{|0}}\left(W\left(F \cdot \varphi_t^\Lambda(\epsilon(r, X)) \cdot \varphi_s^\Theta(\epsilon(r, X)) \cdot \varphi_{-t}^\Lambda\left(\epsilon\left(\varphi_s^{\Theta^\sharp}(r, X)\right)\right)\right)\right)\right) \\ &= -\frac{\partial}{\partial t_{|0}}\left(\frac{\partial}{\partial s_{|0}}\left(W\left(F \cdot \varphi_t^\Lambda(\epsilon(r, X)) \cdot \varphi_s^\Theta(\epsilon(r, X))\right)\right)\right) \end{aligned}$$

for all F. Here, we are using that φ_t^Λ is totally contained in $\Omega(\mathcal{C})$.

Then, by changing F by $G = F \cdot \varphi_l^\Lambda(\epsilon(r, X))$ we have that

$$\begin{aligned} & \frac{\partial}{\partial t_{|l}}\left(\frac{\partial}{\partial s_{|0}}\left(W\left(\varphi_{-t}^\Theta\left(\varphi_s^\Lambda\left(\varphi_t^\Theta(F)\right)\right)\right)\right)\right) \\ & = -\frac{\partial}{\partial t_{|l}}\left(\frac{\partial}{\partial s_{|0}}\left(W\left(F \cdot \varphi_t^\Lambda(\epsilon(r, X)) \cdot \varphi_s^\Theta(\epsilon(r, X))\right)\right)\right) \\ & = -\frac{\partial}{\partial t_{|0}}\left(\frac{\partial}{\partial s_{|0}}\left(W\left(G \cdot \varphi_t^\Lambda(\epsilon(r, X)) \cdot \varphi_s^\Theta(\epsilon(r, X))\right)\right)\right) \\ & = 0 \end{aligned}$$

Therefore,

$$\frac{\partial}{\partial s_{|0}}\left(W\left(\varphi_{-t}^\Theta\left(\varphi_s^\Lambda\left(\varphi_t^\Theta(F)\right)\right)\right)\right) = 0.$$

On the other hand,

$$\begin{aligned} & \frac{\partial}{\partial s_{|l}}\left(W\left(\varphi_{-t}^\Theta\left(\varphi_s^\Lambda\left(\varphi_t^\Theta(F)\right)\right)\right)\right) \\ & = \frac{\partial}{\partial s_{|l}}\left(W\left(F \cdot \varphi_t^\Theta(\epsilon(r, X)) \cdot \varphi_s^\Lambda\left(\epsilon\left(\varphi_t^{\Theta^\sharp}(r, X)\right)\right) \cdot \varphi_{-t}^\Theta\left(\epsilon\left(\varphi_t^{\Theta^\sharp}(r, X)\right)\right)\right)\right) \\ & = \frac{\partial}{\partial s_{|0}}\left(W\left(H \cdot \varphi_t^\Theta(\epsilon(r, X)) \cdot \varphi_s^\Lambda\left(\epsilon\left(\varphi_t^{\Theta^\sharp}(r, X)\right)\right) \cdot \varphi_{-t}^\Theta\left(\epsilon\left(\varphi_t^{\Theta^\sharp}(r, X)\right)\right)\right)\right) \\ & = \frac{\partial}{\partial s_{|0}}\left(W\left(\varphi_{-t}^\Theta\left(\varphi_s^\Lambda\left(\varphi_t^\Theta(H)\right)\right)\right)\right) \\ & = 0 \end{aligned}$$

Where, by simplicity, we are considering the following change of notation,

$$H = F \cdot \varphi_t^{\Theta}\left(\epsilon\left(r, X\right)\right) \cdot \varphi_l^{\Lambda}\left(\epsilon\left(\varphi_t^{\Theta^\sharp}\left(r, X\right)\right)\right) \cdot \varphi_{-t}^{\Theta}\left(\epsilon\left(\varphi_t^{\Theta^\sharp}\left(r, X\right)\right)\right)$$

There, we have already proved that,

$$W\left(\varphi_{-t}^{\Theta}\left(\varphi_s^{\Lambda}\left(\varphi_t^{\Theta}\left(F\right)\right)\right)\right) \equiv Cte.$$

So, evaluating at $t = s = 0$,

$$W\left(\varphi_{-t}^{\Theta}\left(\varphi_s^{\Lambda}\left(\varphi_t^{\Theta}\left(F\right)\right)\right)\right) \equiv W\left(F\right),\ \forall F.$$

Hence, Eq. (26) is satisfied. □

As we promised, this result will give us a computational way of studying the morphogenesis property. Let be a local system of coordinates $\left(t, s, x^i, y^j, y^i_j\right)$ on $\Phi\left(\mathcal{V}\right)$. Consider a left-invariant vector field Θ on $\Phi\left(\mathcal{V}\right)$ and a left-invariant vector field Λ, tangent to the α fibres, on $\Phi\left(\mathcal{V}\right)$,

$$\Theta\left(t, s, x^i, y^j, y^i_j\right) = \lambda\frac{\partial}{\partial t} + \Theta^i\frac{\partial}{\partial x^i} + y^i_l\Theta^l_j\frac{\partial}{\partial y^i_j}, \qquad \Lambda\left(t, s, x^i, y^j, y^i_j\right) = y^i_l\Lambda^l_j\frac{\partial}{\partial y^i_j} \tag{29}$$

Therefore,

$$[\Theta, \Lambda] = y^i_l\left(\Theta^l_r\Lambda^r_j + -\Lambda^l_r\Theta^r_j + \Theta^k\frac{\partial\Lambda^l_j}{\partial x^k} + \lambda\frac{\partial\Lambda^l_j}{\partial t}\right)\frac{\partial}{\partial y^i_j}$$

Then, by using the evolution equation (14), the equation to be satisfied is,

$$y^i_l\left(\Theta^l_r\Lambda^r_j + \Theta^k\frac{\partial\Lambda^l_j}{\partial x^k} + \lambda\frac{\partial\Lambda^l_j}{\partial t} - \Lambda^l_r\Theta^r_j\right)\frac{\partial W}{\partial y^i_j} = 0 \tag{30}$$

for any matrix function $\left(\Lambda^i_j\right)$ fulfilling the equation,

$$y^i_l\Lambda^l_j\frac{\partial W}{\partial y^i_j} = 0 \tag{31}$$

Here the functions λ, Θ^i, Θ^j_i and Λ^j_i depend on the variables t and X. In practice, it is enough to solve Eq. (30) for a basis of the space of solutions of Eq. (31). Thus, the extended material distribution is pointwise generated by the (local) functions λ, Θ^i and Θ^j_i satisfying the Eq. (30) for all functions Λ^j_i satisfying Eq. (31). This equation will be called the **morphogenesis equation** by virtue of Proposition 17.

Analogously, we will consider the associated characteristic distribution $A\mathcal{N}_X\left(\mathbb{R}\right)^T$ to the extended material groupoid of $\mathcal{C}$, which will be called *X-extended material distribution*. The base-characteristic distribution $A\mathcal{N}_X\left(\mathbb{R}\right)^\sharp$ will be called *X-extended body-material distribution*. The foliations associated to the X-extended material distribution and the X-extended body-material distribution will be called *extended material foliation* and *extended body-material foliation* and they will be denoted by $\mathcal{NF}_X$ and $\overline{\mathcal{NF}}_X$, respectively.

Theorem 13. *For each material particle $X \in \mathcal{B}$ and each instant t, there exists a transitive Lie subgroupoid $\mathcal{N}_X\left(t\right)$ of $\mathbb{R} \times \mathbb{R} \times \Pi^1\left(\mathcal{B}, \mathcal{B}\right)^X_X$ with base $\mathcal{NF}_X\left(t\right)$, which is totally contained in $\mathcal{N}_X\left(\mathbb{R}\right)$. Furthermore, $\mathcal{NF}_X$ is the coarsest foliation of $\mathbb{R}$ satisfying this property.*

In other words, we have proved the following:

> *For each particle X, the time-line $\mathbb{R}$ may be canonically divided, in a maximal way, into submanifolds (intervals or points given by the foliation $\mathcal{NF}_X$) in such a way that all the leaves are differentiable processes of evolution without morphogenesis of X.*

Therefore, to calculate this foliation permits us to now, exactly, in what instants of time differentiable process of morphogenesis are being produced.

Analogously to Theorem 12, we have the following result:

Proposition 14. *Let $\mathcal{C}$ be a body-time manifold and $\mathcal{N}_X(\mathbb{R})$ the X-extended material groupoid. The extended material distribution $A\mathcal{N}_X(\mathbb{R})^T$ is pointwise generated by the left-invariant vector fields Θ on $\mathbb{R}\times\mathbb{R}\times\Pi^1(\mathcal{B},\mathcal{B})^X_X$ satisfying*

$$TW_X([\Theta,\Lambda])=0, \tag{32}$$

for all left invariant admissible vector fields Λ for the pair $\left(\mathbb{R}\times\mathbb{R}\times\Pi^1(\mathcal{B},\mathcal{B})^X_X,\Omega_X(\mathbb{R})\right)$ tangent to the α-fibres.

Proof. It may be proved analogously to Proposition 12. □

Let be a local system of coordinates $\left(t,s,y^i_j\right)$ on $\mathbb{R}\times\mathbb{R}\times\Pi^1(\mathcal{B},\mathcal{B})^X_X$. Consider a left-invariant vector field Θ on $\mathbb{R}\times\mathbb{R}\times\Pi^1(\mathcal{B},\mathcal{B})^X_X$ and a left-invariant vector field Λ on $\mathbb{R}\times\mathbb{R}\times\Pi^1(\mathcal{B},\mathcal{B})^X_X$, that is

$$\Theta\left(t,s,y^i_j\right)=\lambda\frac{\partial}{\partial t}+\Theta^i_j\frac{\partial}{\partial y^i_j},\qquad \Lambda\left(t,s,y^i_j\right)=\Lambda^i_j\frac{\partial}{\partial y^i_j}\,. \tag{33}$$

Therefore,

$$[\Theta,\Lambda]=y^i_l\left(\Theta^l_r\Lambda^r_j+\lambda\frac{\partial\Lambda^l_j}{\partial t}-\Lambda^l_r\Theta^r_j\right)\frac{\partial}{\partial y^i_j}$$

Hence, by taking into account the evolution equation for X (17), the equation to be satisfied is,

$$y^i_l\left(\Theta^l_r\Lambda^r_j++\lambda\frac{\partial\Lambda^l_j}{\partial t}-\Lambda^l_r\Theta^r_j\right)\frac{\partial W_X}{\partial y^i_j}=0 \tag{34}$$

for each matrix function $\left(\Lambda^i_j\right)$ which satisfies,

$$y^i_l\Lambda^l_j\frac{\partial W_X}{\partial y^i_j}=0 \tag{35}$$

It is important to highlight that, in this case, the functions λ, Θ^i, Θ^j_i and Λ^j_i depends on the variable t. Analogously to the previous case, in practice, we will only have to check the Eq. (34), for a basis of the space of solutions of Eq. (35).

Again, the X-extended material distribution is pointwise generated by the (local) functions λ, Θ^i and Θ^j_i satisfying the Eq. (34) for all functions Λ^j_i satisfying Eq. (35). Thus, this equation will be called the **morphogenesis equation for X**.

As last theoretical results of the paper, we will present explicitly how may we use these morphogenesis equations to study the morphogenesis of the evolution.

Proposition 15 *(Global morphogenesis at X). Let be a body-time manifold $\mathcal{C}$ and a material point X. X presents a smooth process without morphogenesis if, and only if,*

$$dim\left(A\mathcal{N}_X(\mathbb{R})^\sharp_t\right)=1.$$

for all instants t, with $A\mathcal{N}_X(\mathbb{R})^\sharp_t$ the fibre of $A\mathcal{N}_X(\mathbb{R})^\sharp$ at t.

Proof. We only have to observe that the imposed condition implies that $\mathcal{N}_X(\mathbb{R})$ is a transitive Lie subgroupoid of $\Phi(\mathcal{V})$. □

In fact, we may generalize the result as follows:

Proposition 16 *(Local morphogenesis at X). Let be a body-time manifold $\mathcal{C}$, a material point X and an instant t. X presents a smooth process without morphogenesis on an interval of time around t if, and only if,*

$$dim\left(A\mathcal{N}_X(\mathbb{R})^\sharp_t\right)=1.$$

with $A\mathcal{N}_X(\mathbb{R})^\sharp_t$ the fibre of $A\mathcal{N}_X(\mathbb{R})^\sharp$ at t.

Thus, to study if the evolution of a particle X is not producing morphogenesis (local or global), we will have to deal with the morphogenesis equation for X (34),

$$y^i_l \left(\Theta^l_r \Lambda^r_j + +\lambda \frac{\partial \Lambda^l_j}{\partial t} - \Lambda^l_r \Theta^r_j \right) \frac{\partial W_X}{\partial y^i_j} = 0$$

for all functions Λ^j_i satisfying Eq. (35), which is the equation whose solutions generate the X-extended material distribution.

In particular, X presents a (local) smooth process without morphogenesis if, and only if, there exists a (local) solution $\left(\lambda, \Theta^i_j\right)$ of the morphogenesis equation for X with $\lambda \neq 0$, for all Λ^i_j satisfying Eq. (35).

Notice that, to prove that a process of aging is not producing morphogenesis, we will need to deal with isomorphisms which are not in the material groupoid, i.e., isomorphisms satisfying Eq. (24). In particular, the differentiability of a process of aging ($\Omega_X(\mathbb{R})$ or $\Omega(\mathcal{C})$ Lie subgroupoids) does not have relation with the differentiability of the choice of the implants g proving that the process is not a morphogenesis, i.e., satisfying Eq. (24).

Proposition 17 *(Global morphogenesis). Let be a body-time manifold $\mathcal{C}$. $\mathcal{C}$ presents a smooth evolution process without morphogenesis if, and only if,*

i) $dim\left(A\Omega(\mathcal{C})^T_{\epsilon((t,X))}\right)$ *is constant respect to* (t, X)

ii) For some X, $dim\left(A\Omega_X(\mathbb{R})^\sharp_t\right) = 0$, *for some* t.

iii) $dim\left(A\mathcal{N}_X(\mathbb{R})^\sharp_t\right) = 1$.

Here, $A\Omega(\mathcal{C})^T_{\epsilon((t,X))}$ *(resp.* $A\Omega_X(\mathbb{R})^\sharp_t$ *and* $A\mathcal{N}_X(\mathbb{R})^\sharp_t$*) is the fibre of* $A\Omega(\mathcal{C})^T$ *(resp.* $A\Omega_X(\mathbb{R})^\sharp$ *and* $A\mathcal{N}_X(\mathbb{R})^\sharp_t$*) at* $\epsilon((t, X))$ *(resp.* t*).*

In this way, to prove that a material evolution present a smooth aging without morphogenesis, we will have to deal with Eq. (16), Eq. (19) and Eq. (30).

6. Conclusions and future work

Using the concept of groupoid, we have developed a simple and complete geometric theory of the phenomenon of material evolution and, in particular, of the so-called morphogenesis. We have also obtained the corresponding linear equations of evolution, making our results computable.

In a future work we wish to deepen in this theory, and also consider more complex situations such as in media with microstructure or composite media.

Data availability

No data was used for the research described in the article.

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